B.Math.Hons.IInd year
IInd Semestral Exam 2004
Algebra IV : B.Sury

Any score of > 100 *will be counted as* 100

- (10 marks)
  Let p be a prime and let $G \leq S_p$ act transitively on $\{1, 2, \cdots, p\}$. Show that any nontrivial normal subgroup N of G must also act transitively.

  OR

  Let G be a finite group which has s conjugacy classes. Use Burnside's counting formula to prove that the cardinality of $\{(x, y) \in G \times G\}$ is s times $O(G)$.

- (15 marks)
  Let A be a (left) semisimple ring. Prove that every irreducible left A-module is A-isomorphic to a simple left ideal.

  OR

  For a finite group G, prove that the dimension of the center of the group algebra $\mathbb{C}[G]$ is the number of conjugacy classes of G.

  OR

  Prove that any semisimple ring has only finitely many simple left ideals upto A-isomorphism.

- (10 marks)
  Prove that elements x, y of a finite group G are conjugate if, and only if, $\chi(x) = \chi(y)$ for all irreducible complex characters χ of G.

- (10 marks)
  Give an example of a reducible representation of a finite group which is not completely reducible.

- (10 marks)
  Let χ be an irreducible character and C be a conjugacy class of a finite

group G. Prove

$$\frac{\chi(C)}{\dim \chi} \in \bar{\mathbb{Z}}.$$

State clearly what you use.

OR

Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of an arbitrary group G and assume that there is some $N > 0$ for which $\rho(g)^N = I$ for all $g \in G$. Use Burnside's lemma to prove that Im ρ is finite.

- (10 marks)
  For finite groups $H \subset G$, prove Frobenius reciprocity theorem for class functions.

  OR

  Let G be a finite group and H be a subgroup. If ρ is a representation of H, define the induced representation $\mathrm{Ind}_H^G \rho$ of G. If ρ is the trivial 1-dimensional representation, determine $\mathrm{Ind}_H^G \rho$.

- (15 marks)
  Prove that the exponential map is a homeomorphism from a neighbourhood of $0 \in M_n(\mathbb{C})$ onto a neighbourhood of I in $GL_n(\mathbb{C})$.

  OR

  Prove that the columns of e^{tA} form a basis for the vector space of solutions of the system of linear equations $\frac{dX}{dt} = AX$.

- (13 marks)
  Define the tangent space at a point of a matrix group. Find the tangent space at I of $SU(n)$ and find its dimension.

  OR

  Show that the conjugacy classes in $SU(2)$ are, apart from, $\{I\}$ and $\{-I\}$, homeomorphic to 2-spheres.

- (15 marks)
  Define the equivalence of continuous Hilbert space representations of a

compact group. Prove that each continuous representation is equivalent to a unitary representation. State clearly all the facts you use.

OR

Let (π, H) be a unitary representation of a compact group G such that H has a cyclic vector. Outline the proof of the fact that π is a Hilbert space direct sum of finite-dimensional representations. State what all you use.